Excitation spectrum of Andreev billiards with a mixed phase space

H. Schomerus and C. W. J. Beenakker
Instituut-Lorentz, Leiden University, P. O. Box 9506, 2300 RA Leiden, The Netherlands
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We present a semiclassical theory for the excitation spectrum of a ballistic quantum dot weakly coupled to a superconductor, for the generic situation that the classical motion gives rise to a phase space containing islands of regularity in a chaotic sea. The density of low-energy excitations is determined by quantum energy scales that are related in a simple way to the morphology of the mixed phase space. An exact quantum mechanical computation for the annular billiard shows good agreement with the semiclassical predictions, in particular for the reduction of the excitation gap when the coupling to the regular regions is maximal.

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The spectral statistics of quantum systems is intimately related to the nature of the corresponding classical dynamics [1]. Two celebrated examples are that chaoticity of the classical dynamics is reflected in the quantum realm by level repulsion while integrability causes level clustering [2]. Recently, confined two-dimensional electron gases (quantum dots) coupled to a superconductor via a ballistic point contact have become a new arena for the study of quantum-classical correspondences [3–6]. Such systems are commonly called Andreev billiards [7], because of the alternation of ballistic motion (as in a conventional billiard) with Andreev reflection [8] at the interface with the superconductor. The proximity of the superconductor causes a depletion of excited states at low energies (proximity effect). It was found [3] that a chaotic Andreev billiard has an excitation gap of the order of the Thouless energy, while an integrable Andreev billiard has no true gap but an approximately linearly vanishing density of states. (The Thouless energy $E_T = g\delta/4\pi$ is the product of the point contact conductance g, in units of $2e^2/h$, and the mean level spacing δ of the isolated billiard.)

Both chaotic and integrable dynamics are atypical. The generic situation is a mixed phase space, with "islands" of regularity separated from chaotic "seas" by impenetrable dynamical barriers. A generally applicable theory for the proximity effect in ballistic systems should address the case of a mixed phase space. In this paper we present such a theory.

In a semiclassical approach we link the excitation spectrum quantitatively and qualitatively to the morphology of non-communicating regions in phase space. Different regions exhibit greatly varying length scales, which also depend sensitively on the position of the point contact. Still, we find that a general relation exists (in terms of effective Thouless energies) between these classical length scales and the corresponding quantum energy scales. The results for integrable and fully chaotic motion are recovered as special cases. For the mixed phase space our main finding is a reduction of the excitation gap below the value E_T of fully chaotic systems. The reduction can be an order of magnitude, as we illustrate by a numerical calculation in the annular billiard [9] shown in Fig. 1.

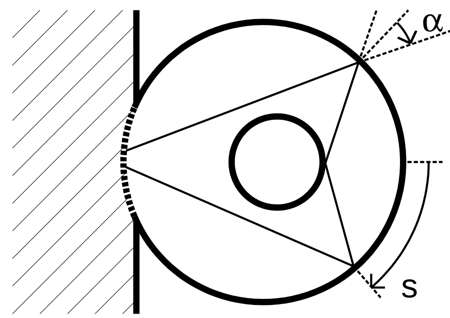

FIG. 1. Andreev billiard consisting of a confined normal conducting region interfacing with a superconductor (shaded) over a distance W. The normal region is shaped like an annular billiard, bounded by two excentric circles (outer radius R, inner radius r, distance of origins ρ). This figure represents the case $R = 1$, $r = 0.35$, $\rho = 0.1$, $W = 0.8$. A periodic trajectory is indicated, involving two Andreev reflections at the interface. For the Poincaré map one monitors the collisions with the outer boundary (angle of incidence α and coordinate s along the boundary, with $\alpha = 0$ denoting normal incidence and $s = 0$ denoting the point closest to the inner circle).

We consider a two-dimensional ballistic region (a "billiard") of area A (mean level spacing $\delta = 2\pi\hbar^2/mA$) that is connected to a superconductor by an opening of width W (corresponding to a dimensionless conductance $g = 2W/\lambda_F$, where λ_F is the Fermi wavelength). Classical trajectories consist of straight lines inside the billiard, with specular reflections at the boundaries and retro-reflections (= Andreev reflections) at the interface with the superconductor. We assume that $\delta \ll E_T \ll \Delta$, where Δ is the excitation gap in the bulk superconductor. The first condition, $\delta \ll E_T$ or $W \gg \lambda_F$, is required for a semiclassical treatment. The second condition, $E_T \ll \Delta$, ensures that the excitation spectrum becomes independent of the properties of the superconductor.

The Andreev billiard has a discrete spectrum for $\varepsilon < \Delta$. (The excitation energy $\varepsilon > 0$ is measured with respect

to the Fermi energy. We count each spin-degenerate level once.) For $\varepsilon \ll \Delta$ the semiclassical expression for the density of states $\rho(\varepsilon)$ reads [3]

$$\rho(\varepsilon) = \frac{2}{\delta} \int_0^\infty \mathrm{d}L\, P(L) \sum_{n=0}^{\infty} \delta\left(\frac{\varepsilon}{2\pi E_T} - \left(n+\tfrac{1}{2}\right)\frac{L_T}{L}\right) , \quad (1)$$

with $P(L)$ the distribution of path lengths between subsequent Andreev reflections. The distribution $P(L)$ is normalised to unity and based on a measure $\mathrm{d}s\,\mathrm{d}\sin\alpha$ for the initial conditions at the interface with the superconductor (see Fig. 1). The length scale $L_T \equiv \hbar v_F/2E_T$ (with v_F the Fermi velocity) is determined by the geometry of the billiard by $L_T = \pi A/W$ and is therefore purely classical. Eq. (1) follows directly from the Bohr-Sommerfeld quantisation rule [3]. The derivation of Lodder and Nazarov [5] starts from the Eilenberger equation [10] for the quasiclassical Green function and arrives at an expression for $\rho(\varepsilon)$ that is mathematically equivalent to Eq. (1).

Return probabilities like $P(L)$ and the related decay of classical correlations have been addressed in many studies [11]. In a chaotic billiard, L_T is the mean path length and $P(L) \propto \exp(-L/L_T)$ is an exponential distribution. Eq. (1) then gives the density of states

$$\rho(\varepsilon) = \frac{2x^2}{\delta}\frac{\cosh x}{\sinh^2 x} , \quad x = \frac{\pi E_T}{\varepsilon} , \quad (2)$$

which drops from $2/\delta$ (the factor of two arises because both electron and hole excitations contribute at positive ε) to exponentially small values as ε drops below $\approx 0.5\,E_T$. Eq. (2) is close to the exact quantum mechanical result [3], which has $\rho \equiv 0$ for $\varepsilon \leq 0.6\,E_T$. For integrable motion $P(L)$ decays algebraically $\propto L^{-p}$ with p close to 3. Eq. (1) then gives $\rho(\varepsilon) \propto \varepsilon^{p-2}$, hence an approximately linearly vanishing density of states. Numerical studies on the circular and rectangular billiard confirm the validity of the semiclassical approach [3,12].

The semiclassical expression (1) holds also for mixed dynamics. It allows to regard each non-communicating region in phase space as a distinct system, to be labelled by an index i. It is helpful to rewrite Eq. (1) in terms of an effective level spacing δ_i and Thouless energy $E_{T,i} \equiv \hbar v_F/2L_{T,i}$ for each of these regions. (This approach extends the Berry-Robnik conjecture [13] to open systems.) We decompose $\rho = \sum_i \rho_i$ into partial densities of states ρ_i, defined by

$$\rho_i(\varepsilon) = \frac{2}{\delta_i} \int_0^\infty \mathrm{d}L\, P_i(L) \sum_{n=0}^{\infty} \delta\left(\frac{\varepsilon}{2\pi E_{T,i}} - \left(n+\tfrac{1}{2}\right)\frac{L_{T,i}}{L}\right) . \quad (3)$$

The distribution $P_i(L)$ (still normalised to unity) now pertains to initial conditions (still with measure $\mathrm{d}s\,\mathrm{d}\sin\alpha$) on the interface with the superconductor that evolve into the i-th region of phase space. On the scale $L_{T,i}$, the distribution $P_i(L)$ decays exponentially for chaotic parts of phase space while algebraic decay is found for regular regions [14]. In each case the partial density of states ρ_i rises to a value $2/\delta_i$ on an energy scale $E_{T,i}$, but while ρ_i has an excitation gap for the chaotic regions it rises linearly for the regular regions. Eq. (3) applies to those regions that are accessible for a given location of the interface. We call these "connected" regions. The other "disconnected" regions (usually some of the regular islands) do not feel the proximity of the superconductor and give a constant background contribution $\rho_i(\varepsilon) = 2/\delta_i$ in the semiclassical approximation.

Two phase space measures O_i and V_i determine the mean length $L_{T,i} = V_i/O_i$ between Andreev reflections, the effective level spacing $\delta_i = (2\pi\hbar)^2/mV_i$, and the effective Thouless energy $E_{T,i} = \hbar v_F/2L_{T,i} = g_i\delta_i/4\pi$, where $g_i = O_i/\lambda_F$ is the effective dimensionless conductance. The first is the area O_i that the region overlaps with the superconducting interface on the Poincaré map (see Fig. 2). It is a measure of the coupling strength of a region to the superconductor. The second is the volume V_i that the region fills out in the full phase space $(\mathbf{r}, \phi)$, where the coordinate $\mathbf{r}$ extends over the area of the billiard and $\phi \in [0, 2\pi)$ is the direction of momentum.

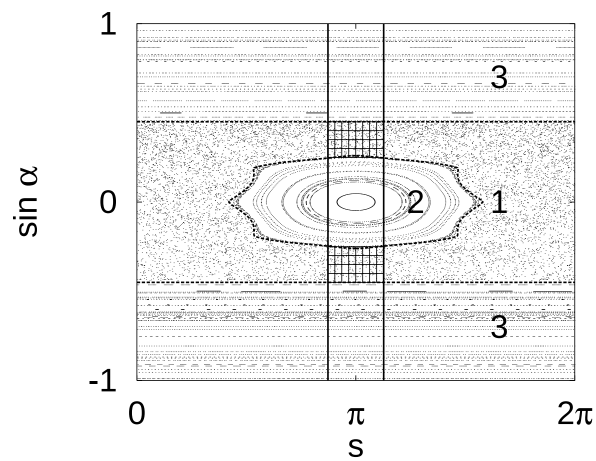

FIG. 2. Poincaré map of the annular billiard of Fig. 1. Dynamical barriers separating regions in phase space are shown as dashed lines. Chaotic trajectories are found in region 1. Region 2 is an island of regular motion around a short stable periodic orbit. Region 3 is integrable and consists of skipping orbits that never hit the inner circle. Initial conditions at the interface with the superconductor are uniformly distributed within the strip around $s = \pi$, of area $2W$. The hatched area is the overlap O_1 of this strip with region 1.

The phase space that is explored from the point contact can also be parameterised by the variables s, $\sin\alpha$ on the interface and a coordinate l along the trajectory. The identification of $L_{T,i} = V_i/O_i$ as the mean path length in region i is a consequence of $\mathrm{d}s\,\mathrm{d}\sin\alpha\,\mathrm{d}l = \mathrm{d}\mathbf{r}\,\mathrm{d}\phi$. The mean length of all trajectories $\langle L\rangle \equiv \int \mathrm{d}L\, LP(L) = \sum_i{}' O_i L_{T,i}/2W = \sum_i{}' V_i/2W$ can be used to determine

the total phase space volume $V_{\rm con} \equiv \sum_i{}' V_i = 2W\langle L\rangle$ that is connected to the interface with the superconductor. Here the prime denotes restriction of the sum to connected regions, and we used the sum rule $\sum_i{}' O_i = 2W$. The volume $V_{\rm dis}$ of the disconnected regions (which determines the background contribution to ρ) follows from the sum rule $\sum_i V_i = V_{\rm con} + V_{\rm dis} = 2\pi A$.

Since typically the smallest $E_{T,i} \ll E_T$, the total density of states $\rho = \sum_i \rho_i$ has a reduced excitation gap. This is especially the case when one couples maximally to the regular regions. Then their contribution to ρ at small ε (long path lengths) is minimal, and the gap is substantially reduced due to long chaotic trajectories. The constant background and the linear increase from regular regions dominates when the coupling is mainly to the chaotic parts of phase space.

The preceding paragraph summarises the key finding of our work. We illustrate it now for the annular billiard of Fig. 1. The Poincaré map in Fig. 2 shows three main regions [15], one with chaotic motion (1) and two with regular motion (2 and 3). The regular island 2 corresponds to orbits that bounce back and forth between the two circles where their distance is largest. It has a short stable periodic orbit in its centre. Region 3 is integrable and consists of skipping orbits (trajectories that do not hit the inner circle, so that their angular momentum $\sin\alpha$ is conserved).

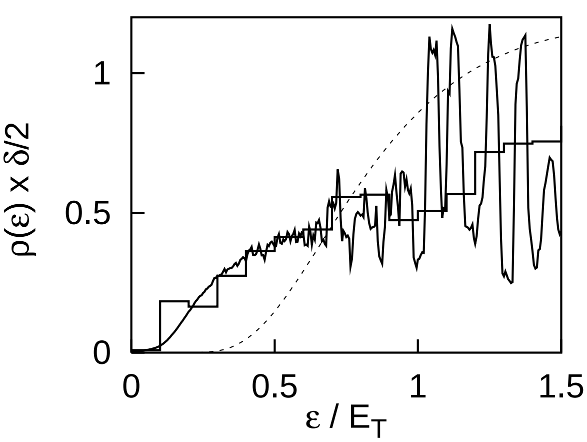

FIG. 3. Density of states of the annular billiard of Fig. 1. The solid curve is the semiclassical prediction computed from Eq. (1). The histogram is obtained by an exact quantum mechanical computation. The dashed curve is the semiclassical result (2) for completely chaotic dynamics.

The regular regions couple maximally to the point contact when it is located at the short stable periodic orbit, as in Fig. 1 (location $s = \pi$). We have computed $P(L)$ by following trajectories and obtained $\rho(\varepsilon)$ from Eq. (1). The result is shown in Fig. 3 (solid curve). We see an excitation gap which is about a factor of 4 smaller than in the fully chaotic case [Eq. (2), dashed curve]. The reduction originates from long chaotic trajectories with mean length $L_{T,1} \approx 4L_T$, hence $E_{T,1} \approx E_T/4$. An exact quantum mechanical calculation [16] (histogram) confirms the low-ε behaviour found semiclassically. The sharp peaks at higher ε in the semiclassical prediction, which arise from families of regular trajectories of almost identical length, are not resolved in the histogram. This is not a surprise since numerically accessible Fermi wavelengths are still larger than the extension of the families.

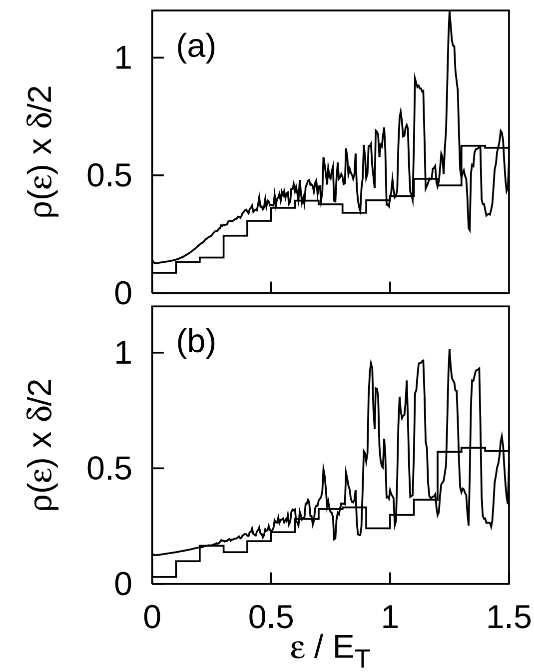

FIG. 4. Density of states of the billiard of Fig. 1, but with two different locations of the interface [$s = 0$ in (a) and $s = 1$ in (b)]. The semiclassical prediction from Eq. (1) (solid curves) is compared with an exact quantum mechanical computation (histograms).

The regular island is disconnected from the superconductor when the point contact is moved to the other end of the billiard (at $s = 0$, where the separation of the circles is smallest). The gap in the chaotic partial density of states is reduced to a lesser degree than before, see Fig. 4(a). Excitations localised in the regular island give a constant background contribution $2/\delta_2 = 2mV_2/(2\pi\hbar)^2$. If the point contact is placed between these two extreme positions (at $s = 1$), the regular regions of phase space dominate the low-energy behaviour of $\rho(\varepsilon)$. Instead of an excitation gap we observe a smoothly and slowly increasing density of states, see Fig. 4(b). The histograms in Fig. 4 fall systematically below the semiclassical prediction. We attribute this discrepancy to the constant background contribution in the semiclassical result, which should vanish at small ε because of quantum mechanical tunnelling through the dynamical barrier between regions 1 and 2. This source of error is absent in Fig. 3, because there all regions are directly coupled to the superconductor.

In summary, we have found that the superconductor proximity effect in ballistic systems depends sensitively on the morphology of the classical phase space. The excitation spectrum at low energies can be described in an intuitively appealing way by means of effective Thouless energies and level spacings for the regular and chaotic regions of phase space. If the coupling to the regular regions is maximal, the excitation spectrum exhibits an excitation gap that is much smaller than the gap of a fully chaotic system. Measurement of such a reduced gap would provide a unique insight into the effect of a mixed classical phase space on superconductivity.

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[15] The chaotic region can be subdivided further into three parts separated by weakly impenetrable dynamical barriers (penetration time larger than L_T/v_F). The qualitative description is improved if one assigns to each part its own set of effective quantities $E_{T,i}$, δ_i.

[16] The histograms were obtained by averaging over 250 independent exact spectra (variation of Fermi energy E_F at fixed shape). To find the spectra we computed the scattering matrices $S(E_F \pm \varepsilon)$ in incremental steps of ε much smaller than a mean level spacing. An excitation is encountered when an eigenvalue of $S(E_F+\varepsilon)S^*(E_F-\varepsilon)$ passes through the point -1 on the unit circle. See: C. W. J. Beenakker, Phys. Rev. Lett. **67**, 3836 (1991).